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Approximation formulas for Landau's constants

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ABSTRACT

In this paper, we establish approximation formulas for evaluating Landau's constants.

1. Introduction

Landau's constants are defined for all integers $n \geqslant 0$ by

$$G_n = \sum_{k=0}^{n} \frac{1}{16^k} \binom{2k}{k}^2$$

and play an important role in complex analysis.

Landau studied the asymptotic behavior of G_n and showed that $G_n \sim (1/\pi)\ln n$. Watson [13] continued this investigation and proved the asymptotic expansion

$$G_n = \frac{1}{\pi}\ln(n+1) + c_0 - \frac{1}{4\pi(n+1)} + O\left(\frac{1}{n^2}\right) \quad (n \to \infty), \tag{1}$$

where

$$c_0 = \frac{1}{\pi}(\gamma + 4\ln 2) = 1.06627\ldots. \tag{2}$$

Here $\gamma = 0.57721\ldots$ denotes Euler's constant. In what follows, c_0 is given in (2). Inspired by formula (1), Brutman [2] discovered upper and lower bounds for G_n:

$$1 + \frac{1}{\pi}\ln(n+1) \leqslant G_n < c_0 + \frac{1}{\pi}\ln(n+1) \quad (n \geqslant 0). \tag{3}$$

New bounds for G_n were given by Falaleev [4], who proved

$$c_0 + \frac{1}{\pi}\ln\left(n + \frac{3}{4}\right) < G_n \leqslant 1.0976 + \frac{1}{\pi}\ln\left(n + \frac{3}{4}\right) \quad (n \geqslant 0). \tag{4}$$

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We easily see from the asymptotic (1) and Theorem 2 below that Falaleev's approximation

$$G_n \approx c_0 + \frac{1}{\pi}\ln\left(n+\frac{3}{4}\right)$$

is much better than Brutman's approximation

$$G_n \approx c_0 + \frac{1}{\pi}\ln(n+1),$$

since

$$G_n - c_0 - \frac{1}{\pi}\ln\left(n+\frac{3}{4}\right) = O\left(\frac{1}{n^2}\right) \quad \text{and} \quad G_n - c_0 - \frac{1}{\pi}\ln(n+1) = O\left(\frac{1}{n}\right).$$

Cvijović and Klinowski [3] provided upper and lower bounds for G_n in terms of the logarithmic derivative of the gamma function, $\psi = \Gamma'/\Gamma$:

$$c_0 + \frac{1}{\pi}\psi\left(n+\frac{5}{4}\right) < G_n < 1.0725 + \frac{1}{\pi}\psi\left(n+\frac{5}{4}\right) \quad (n \geqslant 0), \tag{5}$$

$$0.9883 + \frac{1}{\pi}\psi\left(n+\frac{3}{2}\right) < G_n < c_0 + \frac{1}{\pi}\psi\left(n+\frac{3}{2}\right) \quad (n \geqslant 0). \tag{6}$$

Alzer [1] established sharp inequalities for G_n in terms of the psi function:

$$c_0 + \frac{1}{\pi}\psi(n+\alpha) < G_n \leqslant c_0 + \frac{1}{\pi}\psi(n+\beta) \quad (n \geqslant 0) \tag{7}$$

with the best possible constants

$$\alpha = \frac{5}{4} \quad \text{and} \quad \beta = 1.26621\ldots.$$

Alzer [1, Remark 1] pointed out that for all integers $n \geqslant 1$, the upper and lower bounds for G_n given in (7) improve the bounds presented in (3)–(6).

Recently, Zhao [14, Theorem 1] and Popa [12] established sharp inequalities for G_n such that imply the above Watson asymptotic formulas.

The aim of this paper is to establish approximation formulas for evaluating Landau's constants.

2. Main results

In view of (1), we define the sequence $(u_n)_{n\in\mathbb{N}}$ by

$$u_n = G_n - \frac{1}{\pi}\ln(n+a) - c_0 + \frac{b}{n+c}. \tag{8}$$

We are interested in finding the values of the parameters a, b and c such that $(u_n)_{n\in\mathbb{N}}$ is the *fastest* sequence which would converge to zero. This provides the best approximations of the form:

$$G_n \approx \frac{1}{\pi}\ln(n+a) + c_0 - \frac{b}{n+c}. \tag{9}$$

Our study is based on the following lemma.

Lemma 1. *If the sequence $(\lambda_n)_{n\in\mathbb{N}}$ converges to zero and if there exists the following limit*:

$$\lim_{n\to\infty} n^k(\lambda_n - \lambda_{n+1}) = l \in \mathbb{R} \quad (k > 1),$$

then

$$\lim_{n\to\infty} n^{k-1}\lambda_n = \frac{l}{k-1} \quad (k > 1).$$

This lemma is suitable for accelerating some convergences, or in constructing some asymptotic expansions. For proofs and other details, see, e.g. [5–11].

Theorem 1. *Let the sequence $(u_n)_{n\in\mathbb{N}}$ be defined by* (8). *Then for*

$$\begin{cases} a = \dfrac{3}{4} + \dfrac{\sqrt{198+132\sqrt{3}}}{24}, \\ b = \dfrac{\sqrt{198+132\sqrt{3}}}{24\pi}, \\ c = \dfrac{3}{4} + \dfrac{\sqrt{3}\sqrt{198+132\sqrt{3}}}{72}, \end{cases} \tag{10}$$

or

$$\begin{cases} a = \dfrac{3}{4} - \dfrac{\sqrt{198+132\sqrt{3}}}{24}, \\ b = -\dfrac{\sqrt{198+132\sqrt{3}}}{24\pi}, \\ c = \dfrac{3}{4} - \dfrac{\sqrt{3}\sqrt{198+132\sqrt{3}}}{72}, \end{cases} \tag{11}$$

we have

$$\lim_{n\to\infty} n^5(u_n - u_{n+1}) = \frac{4281+9680\sqrt{3}}{276480\pi} \quad \textit{and} \quad \lim_{n\to\infty} n^4 u_n = \frac{4281+9680\sqrt{3}}{1105920\pi}. \tag{12}$$

The speed of convergence of the sequence $(u_n)_{n\in\mathbb{N}}$ is given by the order estimate $O(n^{-4})$.

Proof. Applying the representation [1, p. 218]

$$G_n - G_{n-1} = \frac{(\Gamma(2n+1))^2}{16^n(\Gamma(n+1))^4},$$

we have

$$u_n - u_{n+1} = -\frac{(\Gamma(2n+3))^2}{16^{n+1}(\Gamma(n+2))^4} - \frac{1}{\pi}\ln(n+a) + \frac{b}{n+c} + \frac{1}{\pi}\ln(n+1+a) - \frac{b}{n+1+c}.$$

We write the difference $u_n - u_{n+1}$ as the following power series in n^{-1}:

$$\begin{aligned} u_n - u_{n+1} = {} & \frac{3-4b+4c\pi}{4\pi n^2} + \frac{-115+96b+96b^2-96c\pi-192cd\pi}{96\pi n^3} \\ & + \frac{203-128b-192b^2-128b^3+128c\pi+384cd\pi+384cd^2\pi}{128\pi n^4} \\ & + \big(-20007+10240b+20480b^2+20480b^3+10240b^4-10240c\pi \\ & - 40960cd\pi - 61440cd^2\pi - 40960cd^3\pi\big)\frac{1}{10240\pi n^5} + O\left(\frac{1}{n^6}\right). \end{aligned} \tag{13}$$

According to Lemma 1, the three parameters a, b and c, which produce the fastest convergence of the sequence $(u_n)_{n\in\mathbb{N}}$ are given by (13)

$$\begin{cases} 3-4b+4c\pi = 0, \\ -115+96b+96b^2-96c\pi-192cd\pi = 0, \\ 203-128b-192b^2-128b^3+128c\pi+384cd\pi+384cd^2\pi = 0, \end{cases}$$

that is, by (10) and (11). We thus find that

$$u_n - u_{n+1} = \frac{4281+9680\sqrt{3}}{276480\pi n^5} + O\left(\frac{1}{n^6}\right).$$

Finally, by using Lemma 1, we obtain assertion (12) of Theorem 1. □

Solutions (10) and (11) provide the best approximations of type (9):

$$G_n \approx \frac{1}{\pi}\ln\left(n + \frac{3}{4} + \frac{\sqrt{198+132\sqrt{3}}}{24}\right) + c_0 - \frac{3\sqrt{198+132\sqrt{3}}}{\pi(72n+54+\sqrt{3}\sqrt{198+132\sqrt{3}})} \tag{14}$$

and

$$G_n \approx \frac{1}{\pi}\ln\left(n+\frac{3}{4}-\frac{\sqrt{198+132\sqrt{3}}}{24}\right)+c_0+\frac{3\sqrt{198+132\sqrt{3}}}{\pi(72n+54-\sqrt{3}\sqrt{198+132\sqrt{3}})}. \tag{15}$$

In view of inequalities (3) and (4), we define the sequence $(v_n)_{n\in\mathbb{N}}$ by

$$v_n = G_n - \frac{1}{\pi}\ln(n+d) - c_0. \tag{16}$$

Following the same method used in the proof of Theorem 1, we can prove the following:

Theorem 2. *Let the sequence $(v_n)_{n\in\mathbb{N}}$ be defined by* (16)*. Then for $d=\frac{3}{4}$, we have*

$$\lim_{n\to\infty} n^3(v_n - v_{n+1}) = \frac{11}{96\pi} \quad \text{and} \quad \lim_{n\to\infty} n^2 v_n = \frac{11}{192\pi}. \tag{17}$$

The speed of convergence of the sequence $(v_n)_{n\in\mathbb{N}}$ is given by the order estimate $O(n^{-2})$.

Motivated by Theorem 2, we define the sequence $(w_n)_{n\in\mathbb{N}}$ by

$$w_n = G_n - \frac{1}{\pi}\ln\left(n+\frac{3}{4}\right) - c_0 - \frac{p}{n^2+qn+r}. \tag{18}$$

Following the same method used in the proof of Theorem 1, we can prove the following:

Theorem 3. *Let the sequence $(w_n)_{n\in\mathbb{N}}$ be defined by* (18)*. Then for*

$$p = \frac{11}{192\pi}, \qquad q = \frac{3}{2}, \qquad r = \frac{5501}{7040}, \tag{19}$$

we have

$$\lim_{n\to\infty} n^7(w_n - w_{n+1}) = \frac{89684299}{3027763200\pi} \quad \text{and} \quad \lim_{n\to\infty} n^6 w_n = \frac{89684299}{18166579200\pi}. \tag{20}$$

The speed of convergence of the sequence $(w_n)_{n\in\mathbb{N}}$ is given by the order estimate $O(n^{-6})$.

In view of the first inequality in (7), we define the sequence $(x_n)_{n\in\mathbb{N}}$ by

$$x_n = G_n - \frac{1}{\pi}\psi\left(n+\frac{5}{4}\right) - c_0. \tag{21}$$

Following the same method used in the proof of Theorem 1, we can prove the following:

Theorem 4. *Let the sequence $(x_n)_{n\in\mathbb{N}}$ be defined by* (21)*. Then we have*

$$\lim_{n\to\infty} n^3(x_n - x_{n+1}) = \frac{1}{32\pi} \quad \text{and} \quad \lim_{n\to\infty} n^2 x_n = \frac{1}{64\pi}. \tag{22}$$

The speed of convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ is given by the order estimate $O(n^{-2})$.

Remark 1. The numerical calculations presented in this work were performed by using the *Maple* software for symbolic computations.

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